Engineering Notes

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A New Approach to the Space-Axis Rotation

Bong Wie*
University of Texas, Austin, Texas

Introduction

THE formulation of the spacecraft attitude dynamics problem involves considerations of kinematics. One scheme for orienting a rigid body to a desired attitude is to successively rotate about the body-fixed axes, which is commonly known as a body-axis rotation. There are 12 sets of Euler angles for successive rotations about unit vectors fixed in the body. As discussed in Ref. 1, it is also possible to bring a rigid body into an arbitrary orientation by performing three successive rotations involving unit vectors fixed in the inertial reference frame. This scheme then provides another 12 sets of Euler angles for the space-axis rotations.

The coordinate transformation matrices for the body-axis and space-axis rotations are intimately related to each other. Twenty-four such matrices have been tabulated in Ref. 1. This Note presents a new approach for deriving such matrices of the space-axis rotation. Using this approach, an intimate relationship between the body-axis and space-axis rotations is directly obtained. Furthermore, this approach provides insights into the understanding of Euler's principal rotation and the space-axis rotation.

This approach is applied first to a simple rotation about an arbitrary axis. An application of this approach to the space-axis rotation is then presented.

A Simple Rotation

In Ref. 1, a simple rotation is defined as "a motion of a rigid body or reference frame B relative to a rigid body or reference frame A is called a simple rotation of B in A if there exists a line L, called an axis of rotation, whose orientation relative to both A and B remains unaltered throughout the motion." Various approaches have been used to develop several different parameterizations of the direction cosine matrix for a simple rotation (e.g., see Refs. 1-3). Almost every formula can be derived in a variety of ways; however, the approach to be taken here demonstrates its simplicity. It also provides insights into the understanding of the intimate relationship between the body-axis and space-axis rotations.

Suppose unit vectors $\hat{a}_i$ and $\hat{b}_i$ $(i=1,2,3)$ are fixed in reference frames or rigid bodies A and B, respectively, that B is subjected to a simple rotation in A, and that $\hat{a}_i=\hat{b}_i$ prior to the rotation. λ and θ are defined in Fig. 1, and λ_i is defined as $\lambda_i \triangleq \lambda\cdot\hat{a}_i \, (i=1,2,3)$.

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*Assistant Professor, Department of Aerospace Engineering and Engineering Mechanics. Member AIAA.

In order to parameterize the direction cosine matrix in terms of λ_i $(i=1,2,3)$ and θ between A and B, the commonly used sequence of Euler angles is used as follows[4]:

1) Rotate the reference frame A, using an orthonomal matrix M of direction cosines, to cause $\hat{a}_1$ axis to be aligned with the chosen direction λ.

2) Rotate around direction λ through an angle θ.

3) Rotate through an inverse matrix M^{-1} until the frame is aligned with the body-fixed axes.

This successive rotation can be represented as

$$[\hat{b}_1\hat{b}_2\hat{b}_3]=[\hat{a}_1\hat{a}_2\hat{a}_3]MC_1(\theta)M^T$$

$$=[\hat{a}_1\hat{a}_2\hat{a}_3]\begin{bmatrix}\lambda_1 & m_1 & n_1\\ \lambda_2 & m_2 & n_2\\ \lambda_3 & m_3 & n_3\end{bmatrix}\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & -\sin\theta\\ 0 & \sin\theta & \cos\theta\end{bmatrix}\begin{bmatrix}\lambda_1 & \lambda_2 & \lambda_3\\ m_1 & m_2 & m_3\\ n_1 & n_2 & n_3\end{bmatrix} \tag{1}$$

where a row-vector representation[1] is used instead of a conventional column-vector representation. The elements of the total transformation matrix $C \triangleq MC_1(\theta)M^T$ become

$$C_{11}=\lambda_1^2+(m_1^2+n_1^2)\cos\theta$$
$$C_{12}=\lambda_1\lambda_2+(m_1m_2+n_1n_2)\cos\theta+(-m_1n_2+m_2n_1)\sin\theta$$
$$C_{13}=\lambda_1\lambda_3+(m_1m_3+n_1n_3)\cos\theta$$
$$+(-m_1n_3+m_3n_1)\sin\theta,\ \text{etc.} \tag{2}$$

Since each element of the direction cosine matrix M is equal to its cofactor in the determinant of the direction cosine matrix, we have

$$\lambda_1=m_2n_3-m_3n_2$$
$$\lambda_2=m_3n_1-m_1n_3$$
$$\lambda_3=m_1n_2-m_2n_1 \tag{3}$$

Because of the orthonormality of the direction cosine matrix M, we also have

$$\lambda_1\lambda_2+m_1m_2+n_1n_2=0$$
$$\lambda_1\lambda_3+m_1m_3+n_1n_3=0$$
$$\lambda_1^2+m_1^2+n_1^2=1,\ \text{etc.} \tag{4}$$

Thus, the elements of C become

$$C_{11}=\cos\theta+\lambda_1^2(1-\cos\theta)$$
$$C_{12}=-\lambda_3\sin\theta+\lambda_1\lambda_2(1-\cos\theta)$$
$$C_{13}=\lambda_2\sin\theta+\lambda_3\lambda_1(1-\cos\theta),\ \text{etc.} \tag{5}$$

which are the parameterization of the direction cosine matrix in terms of λ_i $(i=1,2,3)$ and θ. This representation can also be derived by using various other approaches; however, the objective of this section is to show the simplicity of applying the new approach to simple rotation about an arbitrary axis.

The remainder of this Note presents an application of this approach to the derivation of the transformation matrices of the space-axis rotation.

Space-Axis Rotation

The space-axis and body-axis rotations are defined here as a successive rotation about the space-fixed axes and body-fixed axes, respectively. A very interesting relationship between these different schemes of successive rotations has been discussed in Ref. 1. For example, if B is subjected successively to $\hat{a}_1$, $\hat{a}_2$, and $\hat{a}_3$ rotations of amounts θ_1, θ_2, and θ_3, respectively, then one can achieve the same orientation by successive $\hat{b}_3$, $\hat{b}_2$, and $\hat{b}_1$ rotations of amounts θ_3, θ_2, and θ_1, respectively. This Note shows that using the approach presented in Eq. (1) will naturally results in such an intimate relationship between the space-axis and body-axis rotations, without requiring explicit determinations of those intermediate matrices given in Ref.1.

Consider a space-axis rotation in the sequence $\theta_1\hat{a}_1$, $\theta_2\hat{a}_2$, and $\theta_3\hat{a}_3$ ($\theta_1\hat{a}_1$ means an $\hat{a}_1$ rotation of amount θ_1). The total transformation matrix can be defined as

$$ {}^{A}C^{B} \triangleq {}^{A}C^{\bar{B}}\ {}^{\bar{B}}C^{\bar{\bar{B}}}\ {}^{\bar{\bar{B}}}C^{B} \tag{6} $$

To deal with the $\hat{a}_1$ rotation, let

$$ {}^{A}C^{\bar{B}} \triangleq C_1(\theta_1) \text{ where } C_1(\theta_1) \triangleq \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_1 & -\sin\theta \\ 0 & \sin\theta_1 & \cos\theta_1 \end{bmatrix} \tag{7} $$

Next, to construct a matrix ${}^{\bar{B}}C^{\bar{\bar{B}}}$ that characterizes the $\hat{a}_2$ rotation with the amount θ_2, let us use the approach discussed in the previous section:

$$ {}^{\bar{B}}C^{\bar{\bar{B}}} = {}^{\bar{B}}C^{A}\ C_2(\theta_2)\ {}^{A}C^{\bar{B}} $$

where

$$ C_2(\theta_2) \triangleq \begin{bmatrix} \cos\theta_2 & 0 & \sin\theta_2 \\ 0 & 1 & 0 \\ -\sin\theta_2 & 0 & \cos\theta_2 \end{bmatrix} \tag{8} $$

Combining Eqs. (7) and (8), we get

$$ {}^{A}C^{\bar{\bar{B}}} = {}^{A}C^{\bar{B}}[{}^{\bar{B}}C^{A}\ C_2(\theta_2)\ {}^{A}C^{\bar{B}}] = C_2(\theta_2)C_1(\theta_1) \tag{9} $$

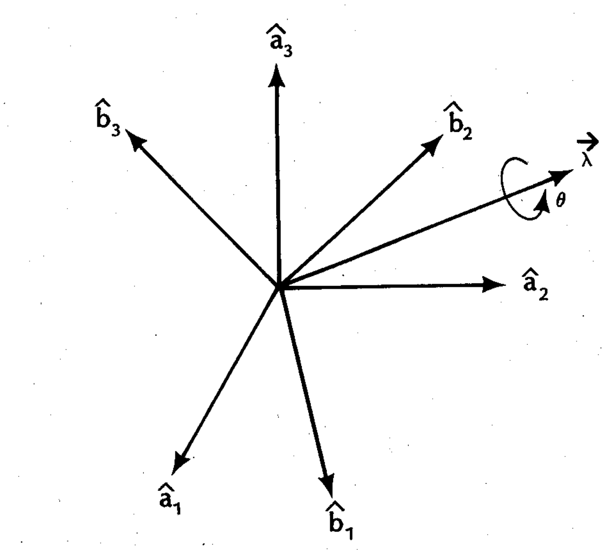

Fig. 1 Geometry of coordinate transformation and Euler's principal rotation.

Similarly, for the $\hat{a}_3$ rotation, we have

$$ {}^{\bar{\bar{B}}}C^{B} = {}^{\bar{\bar{B}}}C^{A}\ C_3(\theta_3)\ {}^{A}C^{\bar{\bar{B}}} $$

where

$$ C_3(\theta_3) \triangleq \begin{bmatrix} \cos\theta_3 & -\sin\theta_3 & 0 \\ \sin\theta_3 & \cos\theta_3 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{10} $$

Finally, the total transformation matrix becomes

$$ {}^{A}C^{B} = {}^{A}C^{\bar{\bar{B}}}[{}^{\bar{\bar{B}}}C^{A}\ C_3(\theta_3)\ {}^{A}C^{\bar{\bar{B}}}] = C_3(\theta_3)C_2(\theta_2)C_1(\theta_1) \tag{11} $$

It can then be noticed that the total transformation matrix represented as Eq. (11) is, indeed, the transformation matrix for the successive $\hat{b}_3$, $\hat{b}_2$, and $\hat{b}_1$ rotations of θ_3, θ_2, and θ_1, respectively. Thus, the total transformation matrix due to the space-axis rotation (successive $\theta_1\hat{a}_1$, $\theta_2\hat{a}_2$, and $\theta_3\hat{a}_3$ rotations) is identical to the transformation matrix due to the body-axis rotation in the sequence of $\theta_3\hat{b}_3$, $\theta_2\hat{b}_2$, and $\theta_1\hat{b}_1$.

Although the total transformation matrix for the space-axis rotation has a simple form as Eq. (11), each intermediate transformation matrix is rather complicated as can be seen from Eqs. (8) and (10). However, the approach used here does not require an explicit determination of those intermediate matrices [e.g., see Eqs. (49) and (52) on p. 36 of Ref. (1)] to find the total transformation matrix. Indeed, an intimate relationship between two different rotation schemes has been obtained rather directly.

Conclusion

A new approach to the derivation of the coordinate transformation matrix for the space-axis rotation has been presented. The use of this approach has naturally resulted in an interesting relationship between the space-axis and body-axis rotations. This approach has also shown its simplicity for parameterizing the direction cosine matrix of the general rotation about an arbitrary axis. However, the practical significance of using the space-axis rotation instead of the commonly used body-axis rotation needs further study.

References

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Closed-Form Solution for a Class of Guidance Laws

Ciann-Dong Yang* and Fang-Bo Yeh†

National Cheng Kung University, Tainan, Taiwan

Introduction

THIS Note presents a closed-form solution of the equations of motion of an ideal missile pursuing a nonmaneuvering target according to a class of guidance laws cur-

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*Instructor, Department of Aeronautics and Astronautics.

†Associate Professor, Institute of Aeronautics and Astronautics.